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S. Aleksić, Z. D. Mitrović and S. Radenović

ON SOME RECENT FIXED POINT RESULTS FOR SINGLE AND MULTI-VALUED MAPPINGS IN b-METRIC SPACES

Abstract. The main purpose of this paper is to improve and correct some results in b-metric spaces. Moreover, we prove that some results can be slightly relaxed and also we explore some proof techniques which provide short proofs of the results.

Key words: b-metric space, b-complete, b-Cauchy, b-continuous, Picard sequence, multi-valued mapping.

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1. Definitions, notations and preliminaries

We start our exposition with the next result which will prove extremely useful in the sequel.

Lemma 1 ([15]). *Let (X, d, s) be a b-metric space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. If there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a b-Cauchy sequence.

Otherwise, for more details on b-metric spaces we refer the reader to ([1]–[5], [8]–[11], [13]–[15], [17]–[21]).

In [6], authors proved some fixed point theorems in b-metric spaces. We will restrict our attention to the following two results.

Theorem 1 ([6] Theorem 1). *Let $(X, d, s \geq 1)$ be a complete b-metric space and define the sequence $\{x_n\}$ in X by the recursion*

$$x_n = Tx_{n-1} = T^n x_0.$$

Let $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 [d(y, Tx) + d(x, Ty)] \tag{1}$$

for all $x, y \in X$, where $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$.

Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point.

Remark 1. The condition $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$ should be replaced by

$$\lambda_i \geq 0,\ i = \overline{1,4}, \qquad \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1.$$

Indeed, for $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_1 = 1$ and $s = 1$, we have

$$d(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$

For $T = I_X$, Theorem 1 is not valid, since the fixed point of T is not unique.

If $s = 1$, then (X, d) is a metric space and the condition (1) implies

$$d(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2} \right\}, \tag{2}$$

where $k = \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$. Note that

$$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 \leq \lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4 < 1.$$

With (2) we recover the well known result for generalized Ćirić contraction map in the metric space and obtain a unique fixed point.

Remark 2. Let us note that the condition $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1$ from [6] implies the condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$. Now, according to the new condition, we can improve the proof of Theorem 1 from [6]. Firstly, the proof that $\{x_n\}$ is a b-Cauchy sequence can be shorter than that in [6]. Indeed, if $x_n \neq x_{n-1}$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(Tx_{n-1}, Tx_n) \leq\ & \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_{n-1}, x_n) + \lambda_3 d(x_n, x_{n+1}) \\ & + \lambda_4 s d(x_{n-1}, x_n) + \lambda_4 s d(x_n, x_{n+1}), \end{aligned}$$

and also

$$\begin{aligned} d(Tx_n, Tx_{n-1}) \leq\ & \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_{n-1}, x_n) \\ & + \lambda_4 s d(x_{n-1}, x_n) + \lambda_4 s d(x_n, x_{n+1}). \end{aligned}$$

It follows easily that $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ where

$$k = \max\left\{\frac{\lambda_1 + \lambda_3 + s\lambda_4}{1 - \lambda_2 - s\lambda_4}, \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}\right\} < 1,$$

and according to Lemma 1, we have that $\{x_n\}$ is a b-Cauchy sequence.

Remark 3. It is not hard to check that the proof of Theorem 1 in [6] is not correct (see pages 3 and 4). Really, the fact that $x^* = \lim\limits_{n\to\infty} x_n$ is the fixed point of T is not clear enough since it was not shown that

$$\frac{s + s^2\lambda_2 + s\lambda_4}{1 - s\lambda_3 - s^2\lambda_4} \qquad \text{and} \qquad \frac{s\lambda_1 + s^2\lambda_2 + s^2\lambda_4}{1 - s\lambda_3 - s^2\lambda_4}$$

are both positive numbers. In fact, for $\lambda_3 = \frac{1}{s}$ we get that both expressions are negative.

We prove this with the new condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$.

$$\begin{aligned}
\frac{1}{s}d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\
&\leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, Tx^*) \\
&\quad + \lambda_4 (d(x^*, x_{n+1}) + d(x_n, Tx^*)) \\
&\leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, Tx^*) \\
&\quad + \lambda_4 d(x^*, x_{n+1}) + \lambda_4 s d(x_n, x^*) + \lambda_4 s d(x^*, Tx^*) \\
&\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) \\
&\quad + \lambda_2 d(x_n, x_{n+1}) + (\lambda_3 + \lambda_4 s) d(x^*, Tx^*),
\end{aligned}$$

or

$$\begin{aligned}
\left(\frac{1}{s} - \lambda_3 - \lambda_4 s\right) d(x^*, Tx^*) &\leq (1 + \lambda_4) d(x^*, x_{n+1}) \\
&\quad + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}).
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
\frac{1}{s}d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx^*, Tx_n) \\
&\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) \\
&\quad + \lambda_3 d(x_n, x_{n+1}) + (\lambda_2 + \lambda_4 s) d(x^*, Tx^*),
\end{aligned}$$

that is.,

$$\begin{aligned}
\left(\frac{1}{s} - \lambda_2 - \lambda_4 s\right) d(x^*, Tx^*) &\leq (1 + \lambda_4) d(x^*, x_{n+1}) \\
&\quad + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_3 d(x_n, x_{n+1}).
\end{aligned} \tag{4}$$

Adding (3) and (4) we obtain

$$\left(\frac{2}{s} - \lambda_2 - \lambda_3 - 2\lambda_4 s\right) d\left(x^*, Tx^*\right)$$
$$\leq 2\left(1+\lambda_4\right) d\left(x^*, x_{n+1}\right) + 2\left(\lambda_1 + \lambda_4 s\right) d\left(x_n, x^*\right)$$
$$+ \left(\lambda_2 + \lambda_3\right) d\left(x_n, x_{n+1}\right) \to 0, \text{ as } n \to \infty.$$

Hence, we can conclude the following.

Conclusion. Theorem 1 from [6] holds if the coefficients $\lambda_i \geq 0$, $i = \overline{1,4}$, satisfy at least one of the following conditions:
1. $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$ for $s \in [1,2]$;
2. $\frac{2}{s} < \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$ for $s \in (2, +\infty)$.

Our approach with the new condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$ provides the generalization and improves Theorem 3.7 from [10], that is., Theorem 2.19 from [18].

Remark 4. Note that condition (2) for $s > 1$ implies

$$d(Tx, Ty) \leq k \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}$$

where $k = \lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4$.

Theorem 2 ([6] Theorem 2)**.** *Let $(X, d, s \geq 1)$ be a complete b-metric space. Let $T : X \to X$ be a mapping for which there exist $\lambda_1, \lambda_2 \in [0, \frac{1}{3})$ such that*

$$d\left(Tx, Ty\right) \leq \lambda_1 d\left(x,y\right) + \lambda_2 \left[d\left(x,Tx\right) + d\left(y,Ty\right)\right], \tag{5}$$

for all $x, y \in X$.

Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is the unique fixed point.

Remark 5. The proof of Theorem 2 is not correct since $\dfrac{\lambda_1 + \lambda_2}{1 - \lambda_2} \leq 1$. We give the improved version of this theorem.

If $s = 1$, then (X, d) is a metric space and the condition $\lambda_1 + 2\lambda_2 < 1$ is appropriate for metric spaces.

Let $s > 1$. By the same method as in [6], we have

$$d(x_n, x_{n+1}) \leq \frac{\lambda_1 + \lambda_2}{1 - \lambda_2} d(x_{n-1}, x_n) = kd(x_{n-1}, x_n).$$

Since $\lambda_1, \lambda_2 \in [0, \frac{1}{3})$, it follows that $\dfrac{\lambda_1 + \lambda_2}{1 - \lambda_2} = k < 1$, and using Lemma 1, we can conclude that the sequence $\{x_n\}$ is a b-Cauchy sequence.

The proof falls naturally into three parts.

Case 1°**.** If T is continuous, then $Tx_n \to Tx^*$ as $n \to \infty$, and $x^* = \lim_{n\to\infty} x_n$ is the fixed point of T.

Case 2°**.** If d is continuous, then substituting $x = x_n$ and $y = \lim_{n\to\infty} x_n$ in (5), we obtain

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \le \lambda_1 d(x_n, x^*) + \lambda_2[d(x_n, x_{n+1}) + d(x^*, Tx^*)].$$

Letting $n \to \infty$, it follows that

$$d(x^*, Tx^*) \le \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \lambda_2 d(x^*, Tx^*),$$

i.e. $(1-\lambda_2)d(x^*, Tx^*) \le 0$. Using the fact that $\lambda_2 \in [0, \frac{1}{3})$, we have $Tx^* = x^*$.

Case 3°**.** Neither 1° nor 2° is satisfied. Then

$$\begin{aligned} \frac{1}{s} d(x^*, Tx^*) &\le d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2[d(x_n, x_{n+1}) + d(x^*, Tx^*)], \end{aligned}$$

i.e. $(\frac{1}{s} - \lambda_2)d(x^*, Tx^*) \le 0$. We conclude that T has a fixed point $x^* = \lim_{n\to\infty} x_n$ if $\lambda_2 < \frac{1}{s}$.

From what has already been proved, we deduce that T has a fixed point if $\lambda_2 < \min\{\frac{1}{3}, \frac{1}{s}\}$.

Now, we will show that our viewpoint sheds some new light on an interesting new result proved in [7].

Theorem 3 ([7] Theorem 2.2)**.** *Let $(X, d, s \ge 1)$ be a complete b-metric space and T, S self-mappings on X which satisfy*

$$\begin{aligned} d(Sx, Ty) \le a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ + a_4 d(y, Sx) + a_5 d(x, y), \end{aligned} \tag{6}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers satisfying:

(i) $s^2 a_1 + s^2 a_2 + s^3 a_3 + s^3 a_4 + s^2 a_5 < 1$,
(ii) $a_1 = a_2$ *or* $a_3 = a_4$.

Then S and T have a unique common fixed point.

Remark 6. If $s = 1$, then (X, d) is a metric space with the assumptions

$$a_1 + a_2 + a_3 + a_4 + a_5 < 1, \quad a_1 = a_2 \quad \text{or} \quad a_3 = a_4.$$

It follows immediately that the condition *(ii)* is superfluous.

We will also prove that [[7] Theorem 2.4] is still true if we drop the assumption of function φ. We repeat the relevant material from [7].

Definition 1. *A function $\psi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if ψ is continuous and strictly increasing and if $\psi(t) = 0$ if and only if $t = 0$.*

Follow the notation used in [7], Φ denotes the next set.

$$\Phi = \Big\{\varphi : [0,\infty)^2 \to [0,\infty) \mid \varphi(0,0) \geq 0,\ \ \varphi(x,y) > 0 \text{ if } (x,y) \neq (0,0),$$
$$\varphi\left(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right) \leq \liminf_{n\to\infty} \varphi(a_n, b_n)\Big\}$$

The next result is stated and proved in [7].

Theorem 4. *Let $(X, d, s \geq 1)$ be a complete b-metric space and T, f self-mappings on X which satisfy*

$$\psi(sd(Tx, fy)) \leq \frac{\psi\left(\frac{d(x,fy)+\frac{d(y,Tx)}{s^3}}{s+1}\right)}{1+\varphi(d(x,fy), d(y,Tx))}, \tag{7}$$

for all $x, y \in X$, where ψ is an altering distance function, $\varphi \in \Phi$ and T is continuous. Then T and f have a unique common fixed point.

Remark 7. We will now show how to dispense with the assumption on function φ. Indeed, the condition (7) implies

$$sd(Tx, fy) \leq \frac{d(x, fy)}{s+1} + \frac{d(y, Tx)}{s^3(s+1)},$$

i.e.

$$\begin{aligned} d(Tx, fy) &\leq \frac{d(x, fy)}{s(s+1)} + \frac{d(y, Tx)}{s^4(s+1)} \\ &\leq \frac{1}{s(s+1)}[d(x, fy) + d(y, Tx)]. \end{aligned}$$

Let $x_0 \in X$, $x_1 = Tx_0$ and $x_2 = fx_1$. Define the sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. It follows that

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(Tx_{2n}, fx_{2n-1}) \\ &\leq \frac{1}{s(s+1)}[d(x_{2n}, fx_{2n-1}) + d(x_{2n-1}, Tx_{2n})] \\ &\leq \frac{1}{s(s+1)}[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})] \\ &\leq \frac{1}{s(s+1)} d(x_{2n-1}, x_{2n+1}) \end{aligned}$$

$$\leq \frac{1}{s+1}[d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})],$$

and we obtain

$$\left(1-\frac{1}{s+1}\right)d(x_{2n+1},x_{2n}) \leq \frac{1}{s+1}d(x_{2n-1},x_{2n}),$$

This clearly forces

$$d(x_{2n+1},x_{2n}) \leq \frac{1}{s}d(x_{2n-1},x_{2n}).$$

According to Lemma 1, we conclude that $\{x_n\}$ is a b-Cauchy sequence.

In the notation of [21], Ψ stands for the family of all functions $\psi,\varphi:[0,\infty)\to[0,\infty)$ with the properties:
(a) $\varphi(t)<\psi(t)$ for each $t>0, \varphi(0)=\psi(0)=0$;
(b) φ and ψ are continuous functions;
(c) ψ is increasing,
and Θ denotes the set of all functions $\theta:[0,\infty)^4\to[0,\infty)$ satisfying the following conditions:
(a) θ is continuous,
(b) $\theta(p,q,r,s)=0$ if and only if $pqrs=0$.

Example 1. The following functions belong to Θ:
1) $\theta(p,q,r,s)=k\min\{p,q,r,s\}+p\cdot q\cdot r\cdot s, \quad k>0$,
2) $\theta(p,q,r,s)=\ln(1+p\cdot q\cdot r\cdot s)$.

Also in [21], a partially ordered set in a b-metric space and a regular space were introduced.

Definition 2 ([21] Definition 2.1)**.** *Let X be a nonempty set. Then $(X,d,\preceq)$ is called a partially ordered b-metric space if d is a b-metric on a partially ordered set $(X,\preceq)$. The space $(X,d,\preceq)$ is called regular if the following condition holds: if a non-decreasing sequence $\{x_n\}$ tends to x, then $x_n\preceq x$ for all n.*

The next theorem is the main result in [21].

Theorem 5. *Suppose that $(X,d,s\geq 1,\preceq)$ is a partially ordered complete b-metric space and $\{T_n\}$ a nondecreasing sequence of self maps on X. If there exists a continuous function $\alpha:X\times X\to[0,1)$ such that for all $x,y\in X$*

$$\alpha(T_ix,T_jy)\leq a_{i,j}\alpha(x,y)$$

and

$$\psi\left(s^3d(T_ix,T_jy)\right)\leq\alpha(x,y)\varphi(M_{i,j}(x,y)) + \theta(d(x,T_ix),d(y,T_jy),d(x,T_jy),d(y,T_ix)),$$

for all $x, y \in X$ *with* $x \preceq y$, *where* $(\psi, \varphi) \in \Psi, \theta \in \Theta$ *and*

$$M_{i,j}(x,y) = \max\left\{d(x,y), d(x,T_i x), d(y,T_j y), \frac{d(x,T_j y) + d(y,T_i x)}{2s}\right\},$$

and $0 \leq a_{i,j}$ $(i,j \in \mathbb{N})$, *satisfy*

(*i*) $A_n = \prod_{i=1}^{n} a_{i,i+1} < 1$, *for all* n,

(*ii*) $\overline{\lim_{i\to\infty}} a_{i,j} < 1$, *for each* j.

Suppose that:

(*i*) T *is continuous, or*

(*ii*) $(X, d, \preceq)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$, *then all* $T_n's$ *have a common fixed point in* X.

Remark 8. The proof of Theorem 5 can be much shorter using Lemma 1. Indeed, on page 59, the proof should start with $\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))$, and it follows that

$$\begin{aligned}
&\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
&\qquad \leq \alpha(x_{n-1}, x_n)\varphi(M_{n,n+1}(x_{n-1}, x_n)) \\
&\qquad\quad + \theta(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\
&\qquad = \alpha(x_{n-1}, x_n)\varphi(M_{n,n+1}(x_{n-1}, x_n)) \\
&\qquad \leq A_{n-1}\alpha(x_0, x_1)\varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).
\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then we have

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
&\leq A_{n-1}\alpha(x_0, x_1)\varphi(d(x_n, x_{n+1})) \\
&\leq A_{n-1}\alpha(x_0, x_1)\psi(d(x_n, x_{n+1})) \\
&< \psi(d(x_n, x_{n+1})),
\end{aligned}$$

which is impossible.

We can conclude that $\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \leq \psi(d(x_n, x_{n+1}))$. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \quad \lambda \leq \frac{1}{s^3}.$$

By Lemma 1, the sequence $\{x_n\}$ is a b-Cauchy sequence and $x_n \to x$ as $n \to +\infty$.

In [13] the authors defined Chatterjea's type contraction in the context of b-metric spaces and proved the following result.

Theorem 6 ([13] Theorem 2.1)**.** *Let $(X, d, s \geq 1)$ be a complete b-metric space, d a continuous function, $T : X \to X$ a Chatterjea's map such that the inequality $\sup_{n\in\mathbb{N}} d(T^n x, x) < \infty$ holds for all $x \in X$. Then*

(i) *there exists a unique fixed point (say ξ) of T;*

(ii) *for any $x_0 \in X$ the sequence $\{x_n\}$ converges to ξ, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$;*

(iii) *there holds the a priori error estimate*

$$d(\xi, T^m x) \leq \alpha^m \sup_{j\in\mathbb{N}} d\left(T^j x, x\right). \tag{8}$$

Recently, C. Chifu and G. Petrusel ([17], Theorem 2.1, Theorem 2.2.) considered the existence of fixed points for some multi-valued mappings in the context of b-metric spaces.

Theorem 7 ([17] Theorem 2.1)**.** *Let $(X, d, s > 1)$ be a complete b-metric space and $T : X \to P(X)$ a multivalued operator such that:*

(i) *there exist $a, b, c \in \mathbb{R}_+$, $a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that*

$$\begin{aligned} H(T(x), T(y)) \leq{} & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned}$$

for all $x, y \in X$;

(ii) *T is closed.*

In these conditions $\mathrm{Fix}(T) \neq \emptyset$.

Theorem 8 ([17] Theorem 2.2)**.** *Let $(X, d, s > 1)$ be a complete b-metric space and $T : X \to P(X)$ a multi-valued operator such that:*

(i) *there exist $a, b, c \in \mathbb{R}_+$, $a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that*

$$\begin{aligned} H(T(x), T(y)) \leq{} & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned}$$

for all $x, y \in X$;

(ii) *T is closed.*

If $S\mathrm{Fix}(T) \neq \emptyset$, then $S\mathrm{Fix}(T) = \mathrm{Fix}(T) = \{x\}$.

Remark 9. Note that we did not really have to use the condition $b+cs < \frac{1}{s}$. Indeed, since $a+b+2cs < \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2} < \frac{1}{s}$, then $b+cs < a+b+2cs < \frac{1}{s}$.

Remark 10. In Theorem 7 and 8, the contractive condition

$$\begin{aligned} H(T(x), T(y)) \leq{} & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned} \tag{9}$$

where $a + b + 2cs < \frac{s-1}{s^2}$, can be replaced by the next two conditions:

$$H(T(x), T(y)) \leq a_1 d(x, y) + b_1 D(x, T(x)) + c_1 D(y, T(y)) + d_1 [D(x, T(y)) + D(y, T(x))], \tag{10}$$

where $a_1 + \frac{b_1+c_1}{2} + 2d_1 s < \frac{s-1}{s^2}$ and

$$H(T(x), T(y)) \leq \lambda_1 d(x, y) + \lambda_2 D(x, T(x)) + \lambda_3 D(y, T(y)) + \lambda_4 D(x, T(y)) + \lambda_5 D(y, T(x)), \tag{11}$$

with $\lambda_1 + \frac{\lambda_2+\lambda_3}{2} + s(\lambda_4 + \lambda_5) < \frac{s-1}{s^2}$.

We will now show that all the above contractive conditions are equivalent to each other.

It is easily seen that (9) implies (10) and from (10) we have (11). Substituting $H(T(y), T(x))$ into (1.11) and combining with (11), we obtain

$$H(T(x), T(y)) \leq \lambda_1 d(x, y) + \frac{\lambda_2 + \lambda_3}{2} [D(x, T(x)) + D(y, T(y))] + \frac{\lambda_4 + \lambda_5}{2} [D(x, T(y)) + D(y, T(x))].$$

When $a = \lambda_1$, $b = \frac{\lambda_2+\lambda_3}{2}$, $c = \frac{\lambda_4+\lambda_5}{2}$, we have (9).

Each of them can be associated with the general conditions which are considered in the metric spaces:

$$H(T(x), T(y)) \leq k_1 \max\left\{d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2s}, \frac{D(x, Ty) + D(y, Tx)}{2s}\right\}, \tag{12}$$

where $k_1 = a + 2bs + 2cs$, and

$$H(T(x), T(y)) \leq k_2 \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\right\}, \tag{13}$$

with $k_2 = a_1 + b_1 + c_1 + 2sd_1$, and also

$$H(T(x), T(y)) \leq k_3 \max\left\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\right\}, \tag{14}$$

where $k_3 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$.

It follows easily that (12) implies (13) and (13) implies (14).

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Suzana Aleksić
University of Kragujevac
Faculty of Science
Department of Mathematics and Informatics
Radoja Domanovića 12
34000 Kragujevac, Serbia
e-mail: suzanasimic@kg.ac.rs

Zoran D. Mitrović
University of Banja Luka
Faculty of Electrical Engineering
Patre 5, 78000 Banja Luka
Bosnia and Herzegovina
e-mail: zoran.mitrovic@etf.unibl.org

Stojan Radenović
King Saud University
College of Science
Department of Mathematics
Riyadh 11451, Saudi Arabia
e-mail: radens@beotel.rs

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